$$f^i_{(k)} = \nabla_j T^{ij}_{(k)} - S^j_{(k)} b_j^{\ i} + x^i_{(k)} = 0,\ f^3_{(k)} = \nabla_i S^i_{(k)} + b_{ij} T^{ij}_{(k)} + {}^1/_2 h^{-1} \delta_{(k)} E_3 (w^{(2)} - w^{(1)}) + x^3_{(k)} = 0$$

and the two equations that hold for the filler

$$\mu_s = {}^2/_3 E_3^{-1} h^3 \nabla_s \nabla_i q^i - E_3^{-1} \nabla_s M^{33}_{(t)} + E_3^{-1} \nabla_s Q^3 - c_{is} q^i + d_{is} Q^i = 0$$

turn out not to be interrelated and lose their physical meaning. Therefore, the set of relations constructed in the previous sections does not allow formal passage to the model of a transversely-soft filler. The set of relationships taking account of temperature effects on the shell that is needed for this case can be constructed using the procedure described in /5/.

REFERENCES

1. PAIMUSHIN V.N. and BOBROV S.N., On buckling modes of sandwich plates and shells with outer layers of homogeneous and bonded materials, Mekhan. Komposit. Materialov, 1, 1985.
2. Strength. Stability. Vibrations. Handbook, 2, Mashinostroenie, Moscow, 1968.
3. GRIGOLYUK E.I. and CHULKOV P.P., Stability and Vibrations of Sandwich Shells. Nauch. Trud. Inst. Mekh. MGU, 27, 1973.
4. PAIMUSHIN V.N. and BOBROV S.N., Critical loads of hinge-supported rectangular plates of symmetric configuration under bilateral compression of one outer layer, Izv. VUZ, Aviats. Tekh., 2, 1985.
5. PAIMUSHIN V.N., Non-linear theory of average bending of sandwich shells with defects in the form of unglued sections, Prikl. Mekhan., 23, 11, 1987.
6. GALIMOV K.Z., Principles of the Non-linear Theory of Thin Shells, Izd. Kazan Univ. Kazan, 1975.
7. AMBARTSUMYAN S.A., General Theory of Anisotropic Shells, Nauka, Moscow, 1974.
8. PISKUNOV V.G. and VERIZHENKO V.E., Linear and Non-linear Problems of Analysing Laminar Structures. Budievel'nik, Kiev, 1986.

Translated by M.D.F.

PMM U.S.S.R., Vol.54, No.1, pp.74-82, 1990
Printed in Great Britain
0021-8928/90 $10.00+0.00

DEFORMATION OF A VISCOELASTIC CYLINDER FASTENED TO A HOUSING UNDER NON-ISOTHERMAL DYNAMIC LOADING*

L.KH. TALYBLY

The state of stress and strain is determined for a hollow long mechanically incompressible viscoelastic cylinder fastened to an elastic shell. Unlike other publications /1, 2/, the case of non-isothermal dynamic loading is examined. The cylinder material is considered to be physically non-linear and a physically linear viscoelastic medium whose mechanical properties depend considerably on the temperature. The temperature field is inhomogeneous and non-stationary. A change in the inner surface of the cylinder with time is allowed during the loading. The results of the solution enable safe working conditions for the structure under consideration to be found for definite temperature, mechanical, and geometric data. Some characteristic graphs of the contact stress as a function of time are presented in the case of instantaneous delivery of heat to the inner and outer cylinder surfaces.

**Prikl. Matem. Mekhan.*, 54, 1, 93-102, 1990

1. *Formulation of the problem.* A full circular thick-walled long cylinder of viscoelastic material, whose mechanical properties depends substantially on the temperature, is fastened along the outer surface to a thin-walled elastic shell. The cylinder is loaded by an internal pressure $p_a(t) = p_a' H(t) + p_a''(t)$ and an external pressure $p_b(t) = p_b' H(t) + p_b''(t)$ and is subjected to the action of a non-stationary inhomogeneous temperature field $T(r, t) = T_1(r) H(t) + T_2(r, t)$ ($H(t)$ is the Heaviside unit function: $H(t) = 0$ for $t \leqslant 0$, and $H(t) = 1$ for $t > 0$). It is assumed that at least one of the quantities p_a', p_b', and $T_1(r)$ independent of the time t is non-zero. Moreover, $p_a''(0) = p_b''(0) = 0$ and $T_2(r, 0) = 0$. It is assumed that the cylinder inner surface can change with time during loading: $a_0 \leqslant a(t) < b$, where a_0 is the cylinder inner radius for $t = 0$ ($a_0 = a(0)$) and b is the outer radius. Plane strain conditions are assumed.

2. *Fundamental relationships and the solution of the problem in the case of a non-linear viscoelastic cylinder.* We will use the following equations, found experimentally in /3/, as the relations between the deviator quantities of the stresses and deformations for a physically non-linear viscoelastic medium with properties that depend very much on temperature:

$$\frac{s_{ij}}{2G_0} = \varphi_1\left(\frac{\varepsilon_u}{v_T}\right) Э_{ij} - \int_0^t R(t-\tau)\,\varphi_2\left(\frac{\varepsilon_u}{v_T}\right) Э_{ij}\, d\tau \qquad (2.1)$$

Here $s_{ij} = \sigma_{ij} - \sigma\delta_{ij}$ is the stress tensor deviator $Э_{ij} = \varepsilon_{ij} - \varepsilon\delta_{ij}$ is the strain tensor deviator, δ_{ij} are Kronecker deltas, $\sigma = \sigma_{ij}\delta_{ij}/3$ is the mean stress. $\varepsilon = \varepsilon_{ij}\delta_{ij}/3$ is the mean strain, and $\varepsilon_u = (^2/_3 Э_{ij} Э_{ij})^{1/2}$ is the strain intensity. The constant G_0 and the function $R(t)$ are the instantaneous shear modulus and relaxation kernel for a certain standard temperature T_s, given in the temperature range of interest. The function $v_T = v_T(T)$ is found experimentally for each material in conformity with its definition /3/. It possesses the following properties: $v_T(T_s) = 1$, $0 < v_T < 1$ for $T > T_s$ and $v_T > 1$ for $T < T_s$. For the standard temperature T_s Eqs.(2.1) reduce to the non-linear viscoelasticity equations obtained /2/ as a result of the quasilinear theory /4/. For the problem under consideration they take the form

$$\frac{\sigma_\varphi - \sigma_r}{2G_0} = \varphi_1\left(\frac{\varepsilon_u}{v_T}\right)(\varepsilon_\varphi - \varepsilon_r) - \int_0^t R(t-\tau)\,\varphi_2\left(\frac{\varepsilon_u}{v_T}\right)(\varepsilon_\varphi - \varepsilon_r)\, d\tau \qquad (2.2)$$

The cylinder material is assumed to be mechanically incompressible

$$\theta = 3\alpha\Delta T \quad (\Delta T = T - T_0) \qquad (2.3)$$

where $\theta = 3\varepsilon$ is the relative change in volume, $\alpha = \alpha(T)$ is the coefficient of linear thermal expansion, and T_0 is the initial temperature at which there are no stresses and strains.

We will consider an unrelated problem; we will assume that the heating of the cylinder due to strain is negligibly small compared with the heating due to heat conduction. In this case the equations of motion, the boundary and initial conditions, and the geometric Cauchy relationships (the temperature field is considered to be known from the solution of the heat conduction problem)

$$\partial\sigma_2/\partial r - (\sigma_\varphi - \sigma_r)/r = \rho\,\partial^2 u/\partial t^2 \qquad (2.4)$$

$$\sigma_r(a, t) = -p_a(t) \qquad (2.5)$$

$$\sigma_r(b, t) = -q(t) = -p_b(t) - \frac{E_* h u(b, t)}{b^2(1-\nu_*^2)} + \frac{E_* h \alpha_* \Delta T(b, t)}{b(1-\nu_*)} - \rho_* h \frac{\partial^2 u(b, t)}{\partial t^2} \qquad (2.6)$$

$$u\,|_{t=0} = \partial u/\partial t\,|_{t=0} = 0 \qquad (2.7)$$

$$\varepsilon_r = \partial u/\partial r, \quad \varepsilon_\varphi = u/r, \quad \varepsilon_z = 0 \qquad (2.8)$$

are added to (2.2) and (2.3).

Here u is the displacement, ρ is the density of the cylinder material, q is the contact pressure, E_*, ν_*, ρ_*, α_* are the elastic modulus, Poisson's ratio, the density, and the coefficient of linear thermal expansion of the shell material, and h is the shell thickness. The boundary condition (2.6) follows from the relationships for a thin-walled shell and the displacement continuity condition on the boundary $r = b$ /2/. As is seen from (2.8), it is assumed that $\varepsilon_z = 0$. We note that the solution of the problem will not be accompanied by additional difficulties if ε_z is taken to be dependent on the time only, in particular $\varepsilon_z = \alpha_* \Delta T(b, t)$.

Taking account of relatonships (2.8) it follows from the compressibility condition (2.3) that

$$u = c(t)/r + rI(r,t) \quad \left(I(r,t) = \frac{3}{r^2}\int_{a(t)}^{r} \alpha r \Delta T(r,t)\,dr\right) \tag{2.9}$$

where $c(t)$ is an unknown function to be determined. Hence

$$\begin{gathered}\varepsilon_r = c(t)/r^2 - I(r,\ t) + 3\alpha\Delta T(r,\ t), \quad \varepsilon_\varphi = c(t)/r^2 + I(r,\ t) \\ \varepsilon_u = (2/\sqrt{3})\,[(c(t)/r^2 + I(r,\ t))^2 + 3(\alpha\Delta T(r,\ t))^2 - 3\alpha\Delta T(r,\ t)(c(t)/r^2 + \\ I(r,\ t))]^{1/2}\end{gathered} \tag{2.10}$$

Taking account of (2.9) and (2.10) we obtain an expression for $\sigma_\varphi - \sigma_r$, from the relationships (2.2) and using it as well as (2.5) and (2.9), we integrate (2.4). We find

$$\begin{gathered}\sigma_r = -p_a(t) + \left(\frac{d^2c(t)}{dt^2} - 3\alpha\bar{\Delta}T(a(t),t)\right)\rho\ln\frac{r}{a(t)} + \rho\int_{a(t)}^{r} r^{-1}J(r,t)\,dr + \\ \Psi_1(r,t,c(t)) - \int_0^t R(t-\tau)\,\Psi_2(r,\tau,c(\tau))\,d\tau\end{gathered} \tag{2.11}$$

where

$$\begin{gathered}J(r,t) = 3\int_{a(t)}^{r} \alpha r\frac{\partial^2\Delta T}{\partial t^2}\,dr, \quad \bar{\Delta}T(a(t),t) = \Delta T(a(t),t)\left(\frac{da(t)}{dt}\right)^2 + \\ \Delta T(a(t),t)\,a(t)\frac{d^2a(t)}{dt^2} + 2a(t)\frac{da(t)}{dt}\frac{\partial(\Delta T(a(t),t)}{\partial t} \\ \Psi_i(r,t,c(t)) = 2G_0\int_{a(t)}^{r}\left[\frac{2c(t)}{r^3} + \frac{2I(r,t) - 3\alpha\Delta T(r,t)}{r}\right]\varphi_i\left(\frac{\varepsilon_u(r,t,c(t))}{v_T(r,t)}\right)dr\ (i=1,2)\end{gathered}$$

Satisfying the boundary condition (2.6) and taking (2.9) into account, we arrive at the relationship

$$\frac{d^2c(t)}{dt^2} + k(t)\,c(t) = f(t) - \frac{1}{\rho_0}\left[\Psi_1(b,t,c(t)) - \int_0^t R(t-\tau)\,\Psi_2(b,\tau,c(\tau))\,d\tau\right] \tag{2.12}$$

$$\begin{gathered}k(t) = \frac{E_* h}{b^3\rho_0(1-\nu_*^2)}, \quad \rho_0 = \rho\ln\frac{b}{a(t)} + \frac{\rho_* h}{b} \\ f(t) = 3\alpha\bar{\Delta}T(a(t),t) + \frac{1}{\rho_0}\left[p_a(t) - p_b(t) + \frac{E_* h\alpha_*\Delta T(b,t)}{b(1-\nu_*)} - \right. \\ \left.\rho\int_{a(t)}^{b} r^{-1}J(r,t)\,dr - \frac{E_* hI(b,t)}{b(1-\nu_*^2)} - \frac{\rho_* hJ(b,t)}{b}\right]\end{gathered}$$

When taking account of (2.9), conditions (2.7) are satisfied if

$$c(t)\,|_{t=0} = (dc(t)/dt)\,|_{t=0} = 0 \tag{2.13}$$

For $\xi \leqslant t$ the relationship (2.12) is obviously valid. We replace t by ξ in (2.12), we multiply the right- and left-hand sides of this equality by $t-\xi$ and integrate with respect to ξ between 0 and t using condition (2.13). We obtain

$$c(t) = f_1(t) + \int_0^t F(t,\tau,c(\tau))\,d\tau \tag{2.14}$$

$$\begin{gathered}f_1(t) = \int_0^t f(\xi)(t-\xi)\,d\xi, \quad F(t,\tau,c(\tau)) = \sum_{i=1}^{3} F_i(t,\tau,c(\tau)) \\ F_1(t,\ \tau,\ c(\tau)) = -(t-\tau)\,k(\tau)\,c(\tau), \quad F_2(t,\ \tau,\ c(\tau)) = \\ -(t-\tau)\,\Psi_1(b,\ \tau,\ c(\tau))/\rho_0 \\ F_3(t,\tau,c(\tau)) = \frac{1}{\rho_0}\int_\tau^t R(\xi-\varepsilon)(t-\xi)\,\Psi_2(b,\tau,c(\tau))\,d\xi\end{gathered}$$

Eq.(2.14) is a non-linear Volterra integral equation of the second kind. For $T(r,\ t) = T_0 = T_s = \text{const}$ and $\varphi_1 = \varphi_2$ it is identical with the equation obtained in /5/ when solving the corresponding isothermal loading problem.

We will use the method of successive approximations to solve (2.14). We take the solution of the equation in the absence of the integral component: $c_0(t) = f_1(t)$ as the zero-th approximation, where $f_1(t)$ is a known function defined above. Any subsequent approximation will be determined by the formula

$$c_k(t) = f_1(t) + \int_0^t F(t, \tau, c_{k-1}(\tau))\, d\tau \quad (k \geqslant 1) \tag{2.15}$$

Therefore, the process of constructing approximations reduces to evaluating quadratures.

3. *Proof of the convergence of the successive approximations.* The sequence of approximations $c_k(t)$ determined by (2.15) converges uniformly in the segment $0 \leqslant t \leqslant t_*$ if the following sufficient conditions are satisfied /6/

$$[F] \leqslant A(t, \tau)[z], \quad ([F] = |F(t, \tau, z_1) - F(t, \tau, z_2)|, \quad [z] = |z_1 - z_2|) \tag{3.1}$$

$$\int_0^{t_*} dt \int_0^t A^2(t, \tau)\, d\tau \leqslant A_0^2(t_*), \quad \left|\int_0^t F(t, \tau, f_1(\tau))\, d\tau\right| \leqslant n(t)$$

$$\int_0^t n^2(\tau)\, d\tau \leqslant N^2(t_*)$$

We will show that if the following obvious conditions are satisfied

$$0 < \varphi_i(\varepsilon_u) \leqslant d_i', \quad |d\varphi_i(\varepsilon_u)/d\varepsilon_u| \leqslant d_i'' \quad (d_i', d_i'' = \text{const}, \ i = 1, 2) \tag{3.2}$$

the functions F and f_1 in the integral Eq.(2.14) satisfy the convergence conditions (3.1).

We will check that the first condition of (3.1) is satisfied. We have

$$[F] \leqslant \sum_{i=1}^{3} [F_i] \tag{3.3}$$

We will estimate each of the components on the right-hand side of (3.3)

$$[F_1] = (t - \tau)\, k(\tau)[z_1] \equiv A_1(t, \tau)[z], \quad A_1(t, \tau) = E_*(t - \tau)\, h/[b^3\rho(1 - \nu_*^2)] \tag{3.4}$$

$$[F_2] = \frac{(t-\tau)[\Psi_1]}{\rho_0} \leqslant \frac{2G_0(t-\tau)[z]}{\rho_0} \left| \int_{a(\tau)}^{b} \left[\frac{d\varphi_1(\varepsilon_u/v_T)}{d(\varepsilon_u/v_T)} \frac{\partial \varepsilon_u(r, \tau, z)}{\partial z} \frac{1}{v_T(r, \tau)} \times \right.\right. \tag{3.5}$$

$$\left.\left. \left(\frac{2z}{r^3} + r^{-1}(2I(r, \tau) - 3\alpha\Delta T(r, \tau))\right) + \frac{2}{r^3} \varphi_1\left(\frac{\varepsilon_u(r, \tau, z)}{v_T(r, \tau)}\right)\right] dr \right|$$

Taking account of Eqs.(2.10) and the inequalities $|\varepsilon_\varphi - \varepsilon_r| / \varepsilon_u \leqslant 3/\sqrt{2}$ we obtain

$$\left|\frac{\partial \varepsilon_u(r, \tau, z)}{\partial z}\right| = \frac{2}{3} \frac{1}{r^2} \frac{|\varepsilon_\varphi(r, \tau, z) - \varepsilon_r(r, \tau, z)|}{\varepsilon_u(r, \tau, z)} \leqslant \frac{\sqrt{2}}{r^2} \tag{3.6}$$

Moreover

$$|z| \leqslant lr + r^2 |I(r, t)| \tag{3.7}$$

(relationships (2.9) and the inequality $|u(r, t)| \leqslant l$ are used).

Taking account of (3.2), (3.6) and (3.7), we obtain from the inequality (3.5)

$$[F_2] \leqslant A_2(t, \tau)[z], \quad A_2(t, \tau) = (t - \tau)\, \zeta_1(\tau) \tag{3.8}$$

$$\zeta_1(\tau) = \frac{2G_0}{\rho_0} \int_{a(\tau)}^{b} r^{-3} \left[\frac{\sqrt{2}\, d_1''}{v_T(r, \tau)} \left(\frac{2l}{r} + 4|I(r, \tau)| + 3|\alpha\Delta T(r, \tau)|\right) + 2d_1'\right] dr$$

Similarly we can estimate

$$[F_3] \leqslant A_3(t, \tau)[z], \quad A_3(t, \tau) = \int_\tau^t R(\xi - \tau)(t - \xi)\, \zeta_2(\xi)\, d\xi \tag{3.9}$$

where $\zeta_2(\xi)$ corresponds to $\zeta_1(\xi)$ on replacing d_1' by d_2' and d_1'' by d_2''.

Using the estimates (3.4), (3.8) and (3.9), we see that the first condition of (3.1) is satisfied, where

$$A(t,\tau) = \sum_{i=1}^{3} A_i(t,\tau) \tag{3.10}$$

We will now check that the second condition of (3.1) is satisfied. We first estimate each of the components on the right-hand side of (3.10)

Since $\rho > 0$ and $a(t) < b$ we have

$$A_1(t,\tau) \leqslant (t-\tau) E_* / [b^2\rho_*(1-\nu_*^2)] \tag{3.11}$$

On satisfying the inequalities

$$|\alpha| \leqslant \alpha_1 = \text{const}, \quad |\Delta T \quad (r,t)| \leqslant L_0 = \text{const}, \quad r^2 - a^2(t) \leqslant r^2, \quad v_T \geqslant 1 \tag{3.12}$$

(the last inequality can always be satisfied by selecting the temperature T_s in the definition of the function v_T /3/), we obtain from (3.8)

$$A_2(t,\tau) \leqslant m_1(t-\tau), \quad m_1 = \frac{2G_0}{\rho_* h}\left(\frac{2\sqrt{2}\,ld''}{3}\,\frac{b^3-a_0^3}{a_0^3 b^2} + \frac{9\sqrt{2}\alpha_1 d'' L_0 + 2d'}{2}\,\frac{b^2-a_0^2}{a_0^2 b}\right) \tag{3.13}$$

$$d' = \max\{d_1', d_2'\}, \quad d'' = \max\{d_1'', d_2''\}$$

Taking account of the inequalities

$$R(t) > 0, \quad \int_0^\infty R(t)\,dt < 1$$

we can estimate $A_3(t,\tau)$ in an analogous manner from (3.9)

$$A_3(t,\tau) \leqslant m_1 \int_\tau^t (t-\xi) R(\xi-\tau)\,d\xi \leqslant m_1(t-\tau) \tag{3.14}$$

Taking the estimates (3.11), (3.13), (3.14) into account in (3.10), we have

$$A(t,\tau) \leqslant (t-\tau)[2m_1 + E_*/[b^2\rho_*(1-\nu_*^2)]] \tag{3.15}$$

Using the inequality (3.15), we see that the second condition in (3.1) is satisfied. We can now check that the third and fourth conditions in (3.1) are satisfied. We write

$$\left|\int_0^t F(t,\tau,f_1(\tau))\,d\tau\right| \leqslant \sum_{i=1}^{3}\int_0^t |F_i(t,\tau,f_1(\tau))|\,d\tau \tag{3.16}$$

and we estimate each components on the right-hand side. We will first estimate the function $f_1(t)$. We use the expressions for $\bar{\Delta} T(a(t),t)$ and $f(t)$ presented above, the inequalities (3.12) and the inequalities

$$|da(t)/dt| \leqslant a_1, \quad |d^2a(t)/dt^2| \leqslant a_2, \quad |\partial T(r,t)/\partial t| \leqslant L_1$$
$$|\partial^2 T(r,t)/\partial t^2| \leqslant L_2, \quad |P_a(t) - p_b(t)| \leqslant p_0 \quad (a_1, a_2, L_1, L_2, p_0 = \text{const})$$

that hold for any $0 \leqslant t \leqslant t_*$, $a(t) \leqslant r \leqslant b$

$$|f_1(t)| \leqslant m_2 t^2/2, \quad m_2 = 3\alpha_1(bL_0a_2 + 2ba_1L_1 + L_0a_1^2) +$$
$$\frac{b}{\rho_* h}\left(p_0 + \frac{E_* h\alpha_* L_0}{b(1-\nu_*)} + \frac{3}{4}\alpha_1 L_2 \rho b^2 + \frac{3\alpha_1 E_* h L_0}{2b(1-\nu_*^2)} + \frac{3}{2}\alpha_1\rho_* hbL_2\right)$$

The following estimates hold here:

$$|F_1(t,\tau,f_1(\tau))| \leqslant m_3\tau^2(t-\tau), \quad |F_i(t,\tau,f_1(\tau))| \leqslant (t-\tau)(m_4\tau^2 + m_5) \quad (i = 1, 2) \tag{3.17}$$

$$m_3 = \frac{E_* m_2}{2\rho_* b^2(1-\nu_*^2)}, \quad m_4 = \frac{G_0 bd' m_2(b^2-a_0^2)}{\rho_* ha_0^2 b^2}, \quad m_5 = \frac{12G_0 bd'\alpha_1 L_1 \ln b/a_0}{\rho_* h}$$

By using the estimates (3.17) and evaluating the integrals on the right-hand side of inequalities (3.16), it can be shown that the third and fourth conditions in (3.1) are also satisfied.

Therefore, all the conditions (3.1) are satisfied and, consequently, the function $c(t)$ can be calculated to any degree of accuracy by using the successive approximations (2.15). After having determined the function $c(t)$, the displacement $u(r,t)$, the strain $\varepsilon_r(r,t)$, $\varepsilon_\varphi(r,t)$, and the stress $\sigma_r(r,t)$ are found from (2.9)-(2.11) respectively. The stress σ_φ is determined from relationships (2.2), and the stress σ_z in terms of σ_r and σ_φ by the relationship of the theory of plane strain of an incompressible material: $\sigma_z = (\sigma_r + \sigma_\varphi)/2$.

4. Another method of constructing the approximations. This method is based on using the solution of the corresponding problem of the linear theory of thermoviscoelasticity. The necessity for this is caused by the fact that the rate of convergence of the processes can be different in different cases, which is not without distinction.

We will represent the function $\varphi_i(\varepsilon_u/v_T)$ in the form $\varphi_i(\varepsilon_u/v_T) = 1 - \eta_i(\varepsilon_u/v_T)$ $(i = 1, 2)$. We assume $a(t) \equiv a_0 = \text{const}$ (in this case there are effective methods of solving problems of linear viscoelasticity theory /1, 2/). Here $\rho_0 = \text{const}$ and $k = \text{const}$. We introduce the notation

$$k_1 = \frac{2G_0}{\rho_0}\frac{b^2 - a_0^2}{a_0^2 b^2}, \quad f_2(t) = f(t) - \frac{2G_0}{\rho_0}\Big[\int_{a_0}^{b} r^{-1}(2I(r,t) - 3\alpha\Delta T(r,t))\,dr -$$

$$\int_0^t R(t-\tau)\int_{a_0}^{b} r^{-1}(2I(r,\tau) - 3\alpha\Delta T(r,\tau))\,dr\,d\tau\Big].$$

$$\chi_i(t, c(t)) = \frac{2G_0}{\rho_0}\int_{a_0}^{b}\Big[\frac{2c(t)}{r^3} + \frac{2I(r,t) - 3\alpha\Delta T(r,t)}{r}\Big]\eta_i\Big(\frac{\varepsilon_u(r,t,c(t))}{v_T(r,t)}\Big)dr \quad (i = 1, 2)$$

The analogue of (2.12) will be

$$\frac{d^2c(t)}{dt^2} + (k_1 + k)c(t) - k_1\int_0^t R(t-\tau)c(\tau)\,d\tau = \tag{4.1}$$

$$f_2(t) + \chi_1(t, c(t)) - \int_0^t R(t-\tau)\chi_2(\tau, c(\tau))\,d\tau$$

This equation is also solved by the method of successive approximations. The solution for $\eta_1^{(0)} = \eta_2^{(0)} = 0$, is taken as the zero-th approximation from which $\chi_1^{(0)} = \chi_2^{(0)} = 0$ follows. This case corresponds to the solution of the thermoviscoelasticity problem for a medium whose properties are independent of the temperature. Determining the function $c^{(0)}(t)$ from (4.1) and (2.13) for $\chi_1^{(0)} = \chi_2^{(0)} = 0$ we find $\varepsilon_u^{(0)}(r, t, c^{(0)}(t))$ by means of (2.10). Hence $\eta_i^{(1)}(\varepsilon_u^{(0)}/v_T)$ become known and from them $\chi_i^{(1)}(t, c^{(0)}(t))$ $(i = 1, 2)$ as well. Taking account of $\chi_i^{(1)}$ in (4.1), we obtain an equation to determine $c^{(1)}(t)$ that is analogous to the equation for the zero-th approximation. Therefore, continuing the procedure, to determine $c(t)$ we have problem (4.1) and (2.13) in each successive approximation, where the right-hand side of (4.1) will be determined by the previous approximation.

The convergence of the successive approximations is proved in this case as in the preceding case since (4.1) is converted to the form (2.14) exactly as in the case mentioned. Conditions analogous to (3.2) should here be extended to the function $\eta_i(\varepsilon_u/v_T)$.

5. Solution in the case of a linear viscoelastic cylinder. We assume that the cylinder material possesses properties of physical linearity. In this case /3/

$$\varphi_i(\varepsilon_u/v_T) = \omega_i(T) \quad (i = 1, 2) \tag{5.1}$$

The functions $\omega_1(T)$ and $\omega_2(T)$ characterize the influence of the temperature on the mechanical properties of the materials under instantaneous loading and on the rheonomic properties, and are determined experimentally for each material /3, 7/. They possess the properties $\omega_i(T_s) = 1$; $0 < \omega_i(T) < 1$ for $T > T_s$ and $\omega_i(T) > 1$ for $T < T_s$ $(i = 1, 2)$.

In the case under consideration relationships (2.3)-(2.8) remain in the same form in which they were used above. We solve problem (2.2), (5.1), (2.3)-(2.8) for $a(t) = a_0 = \text{const}$, $T(r, t) = T_1(r)H(t)$ $(T_1(r) \neq 0$, $T_2(r, t) = 0)$ where we will assume that the temperature field T is measured from the initial temperature T_0. Here $\rho_0 = \text{const}$, $k = \text{const}$, $I(r, t) = I_1(r)H(t)$, where $I_1(r) = 3r^{-2}\int_0^r \alpha r T_1(r)\,dr$. We note that the displacement u and the strain ε_r, ε_φ are determined by (2.9) and (2.10) even in this case, in which we should assume, $a(t) = a_0$, $\Delta T = T = T_1(r)H(t)$. The relationships (2.11) and (2.12) in the case under consideration are converted, respectively, to the form

$$\sigma_r = -p_a(t) + \rho \frac{d^2c(t)}{dt^2} \ln \frac{r}{a_0} + w_1(r)c(t) - w_2(r)\int_0^t R(t-\tau)c(\tau)\,d\tau + g_1(r)H(t) - g_2(r)\int_0^t R(\xi)\,d\xi \tag{5.2}$$

$$\rho_0 \frac{d^2c(t)}{dt^2} + k_2c(t) = \psi(t) + w_2(b)\int_0^t R(t-\tau)c(\tau)\,d\tau \tag{5.3}$$

$$w_i(r) = 4G_0\int_{a_0}^r r^{-3}\omega_i(T)\,dr, \quad g_i(r) = 2G_0\int_{a_0}^r \mu(r)\omega_i(T)\,dr \quad (i=1,2)$$

$$\mu(r) = r^{-1}(2I_1(r) - 3\alpha T_1(r)), \quad k_2 = k\rho_0 + w_1(b), \quad \psi(t) = p_a(t) - p_b(t) + k_3H(t) + g_2(b)\int_0^t R(\xi)\,d\xi, \quad k_3 = k\rho_0(\alpha_* T_1(b)(1+\nu_*) - b^2I_1(b)) - g_1(b)$$

As we see, the problem is to determine the function $c(t)$ that is a solution of (5.3) under the initial conditions (2.13). Before determining $c(t)$ from (5.5), we will examine the case, that is of independent interest, when the cylinder material is linearly elastic. We set $R(t) = 0$ in (5.3) here, after which by using (2.13) we determine $c(t)$

$$c(t) = \frac{\psi_0}{k_2}\left(1 - \cos\sqrt{\frac{k_2}{\rho_0}}\,t\right), \quad \psi_0 = p_a' - p_b' + k_3 \tag{5.4}$$

(it is assumed that $p_a''(t) \equiv 0$, $p_b''(t) \equiv 0$).

Taking (5.4) into account, we find from (5.2)

$$\sigma_r = -p_a' + \frac{\rho}{\rho_0}\psi_0 \ln\left(\frac{r}{a_0}\right)\cos\sqrt{\frac{k_2}{\rho_0}}\,t + w_1(r)\frac{\psi_0}{k_2}\left(1 - \cos\sqrt{\frac{k_2}{\rho_0}}\,t\right) + g_1(r) \tag{5.5}$$

We note that the dependence of the mechanical properties of the cylinder material on the temperature is taken into account in (5.5) in terms of the function $\omega_1(T)$.

We will now examine the question of determining the function $c(t)$ in the case of a linear viscoelastic cylinder material. Taking account of condition (2.13), we apply a Laplace transformation to (5.3), from which we determine

$$\bar{c}(p) = \bar{\psi}(p)/[\rho_0p^2 + k_2 - w_2(b)\bar{R}(p)] \tag{5.6}$$

For certain constraints on $\bar{R}(p)$ the construction of the original of expressions of the type (5.6) is presented in /2/. Without constraining $\bar{R}(p)$ the function $c(t)$ can be found approximately using the well-known Schapery law /8/

$$c(t) = [p\bar{c}(p)]_{p=(1/2)t^{-1}} = 4t^2\psi(t)/(\rho_0 + 4k_2t^2 - 8w_2(b)t^3R(t)) \tag{5.7}$$

Taking (5.7) into account in (5.2), we obtain an approximate analytical expression for the stress σ_r.

We will assume for a qualitative analysis of the solution that the kernel is represented in the form of an exponential function $R(t) = \Lambda\exp(-\lambda t)$ $(\Lambda > 0, \lambda > 0)$. In this case (5.6) is converted to the form

$$\bar{c}(p) = (\lambda + p)\bar{\psi}(p)/P_3(p)$$
$$P_3(p) = \rho_0p^3 + \lambda\rho_0p^2 + k_2p + k_2\lambda - \Lambda w_2(b) \tag{5.8}$$

Let p_1, p_2 and p_3 be the roots of the polynomial $P_3(p)$. Then (5.8) is represented in the form of the sum of simple fractions

$$\bar{c}(p) = \frac{\bar{\psi}(p)}{\rho_0}\sum_{i=1}^3 \frac{A_i}{p-p_i}$$

$$A_1 = (p_1+\lambda)(p_2-p_3)/M, \quad A_2 = (p_2+\lambda)(p_3-p_1)/M, \quad A_3 = (p_3+\lambda)(p_1-p_2)/M$$

$$M = p_1p_2(p_1-p_2) + p_1p_3(p_3-p_1) + p_2p_3(p_2-p_3)$$

the original $c(t)$ will here have the form

$$c(t) = \sum_{i=1}^{3} \frac{A_i}{\rho_0} \int_0^t \exp\left[p_i(t-\tau)\right] \psi(\tau)\, d\tau \tag{5.9}$$

Having determined the function $c(t)$ by means of (5.9), we find the displacement u, and the strain $\varepsilon_r, \varepsilon_\varphi$ by means of (2.9) and (2.10) for $a(t) = a_0$, $\Delta T(r, t) = T_1(r) H(t)$, and the stress σ_r by means of (5.2). Afterwards the stresses σ_φ and σ_z are determined

$$\sigma_\varphi = \sigma_z + r\partial\sigma_r/\partial r - \rho r \partial^2 u/\partial t^2, \quad \sigma_z = (\sigma_z + \sigma_\varphi)/2.$$

In the case of an exponential kernel the solution obtained will be exact.

6. The influence of the temperature field on the change in contact pressure. The change with time of the contact pressure $\sigma_r(b, t)$ is of interest in the problem under consideration. The behaviour of $\sigma_r(b, t)$ under isothermal loading of the cylinder-shell system by pressures $p_b(t) \equiv 0$, $p_a(t) = p_a' H(t)$ and $p_a(t) \equiv 0$, $p_b(t) = p_b' H(t)$ is studied in /9/ when the cylinder properties are described by the simplest equations for a standard linearly viscoelastic body. Unlike /9/, we have considered the influence of the temperature field on the quantity $\sigma_r(b, t)$ on the basis of the solution obtained above. We take $p_a(t) \equiv 0$. $p_b(t) \equiv 0$. We assume that heat is supplied within the cylinder, which instantaneously creates a temperature T_a on its interior surface, that is later maintained constant, while an instantaneously created constant temperature T_b is kept on the outer surface. The cylinder temperature field is here represented in the form

$$T(r, t) = [T_a + (T_b - T_a) \ln(r/a_0)/\ln(b/a_0)]\, H(t)$$

We assume $\alpha = \text{const}$, $T_s = T_a$, $\omega_1 = (T_s/T)^\gamma$. The numerical values of the parameters are given as follows: $\rho_*/\rho = 0.2$, $h/b = 10^{-2}$, $b/a_0 = 3$, $\nu_* = 0.3$, $\alpha_*/\alpha = 10^{-1}$, $T_a/T_b = 10$.

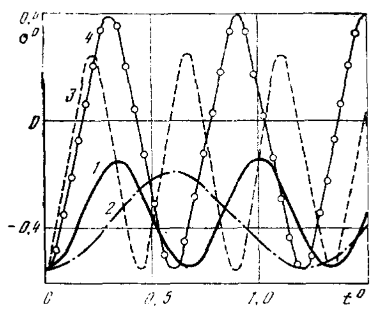

Fig.1

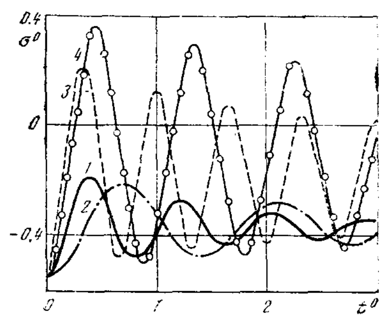

Fig.2

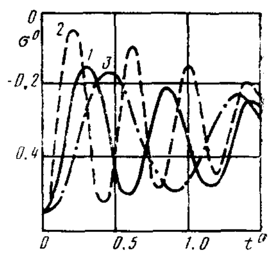

Fig.3

Using these data graphs of the dependence of $\sigma^\circ = \sigma_r(b, t)/(E_* \alpha_* T_b)$ on $t^\circ = (\pi a_0)^{-1} t \sqrt{G_0/\rho}$, are constructed on the basis of (5.5) (the cylinder material is linearly elastic) and represented in Fig.1. Curves *1* and *2* are constructed for $E_*/G_0 = 10^3$ and $\gamma = 2$, $\gamma = 0$ respectively, and curves *3* and *4* for $E_*/G_0 = 10^4$ and $\gamma = 2$, $\gamma = 0$ respectively ($\gamma = 0$ corresponds to the case when the properties of the cylinder material are independent of the temperature). Analogous graphs are presented in Fig.2 in the case when the cylinder material is linearly viscoelastic. We used (5.2) to construct them where (5.9) was taken for the function $c(t)$. In this case, besides the parameter values cited above, we took $\omega_2/\omega_1 = 1/3$, 7/ and $\lambda/\Lambda = 2$, $\sqrt{G_0}/(a_0 \lambda \sqrt{\rho}) = 1$.

As we see, the contact stress $\sigma_r(b, t)$ is compressive for $E_*/G_0 = 10^3$ and the values $\gamma = 0$, $\gamma = 2$. However, when the shell elastic modulus is 10^4 and more times greater than the cylinder shear modulus, the stress $\sigma_r(b, t)$ can become tensile for the same values of γ (tensile stresses are dangerous since they can result in peeling of the shell from the cylinder). Moreover, a sharp increase in the stress amplitudes with respect to the case $\gamma = 2$ is noted in Figs.1 and 2 for $\gamma = 0$ as the values of E_*/G_0 change from 10^3 to 10^4. A phenomenon characteristic for viscoelastic materials is observed in Fig.2: damping of the stress amplitudes with time. It follows from physical considerations that they tend to quasi-static solutions. We note that the damping rate of the stress amplitudes is less in the case when the viscoelastic material properties are independent of the temperature than when such a dependence exists.

Graphs of the contact stress in the case of a non-linear viscoelastic cylinder material, constructed in conformity with the method examined in Sect.4 are represented in Fig.3. The functions $\varphi_1 = \varphi_2 \equiv \varphi$ and v_T were approximated in the form $\varphi(\varepsilon_u) = B\varepsilon_u^\beta$, $v_T = (T_s/T)^\delta$. It was assumed $B = 1$, $\delta\beta = -\gamma$. Curves 1-3 correspond to the following values (β, δ): $(-0.2, 10)$, $(-0.4, 5)$, $(-0.2, 5)$. Results of a third approximation are presented (the results of the zero-th approximation correspond to the case $\gamma = 0$ $(\beta = 0)$ in Fig.2). As we see, the stress amplitudes are increased for times close to zero as the degreee of non-linearity of the material increases (while keeping the values of $\delta\beta$ fixed). Meanwhile, the rate of their damping is increased. The contact stresses do not become tensile for values of the parameters presented in Fig.3, and, consequently, they do not represent a danger for the cylinder-shell system.

Finally, we note that all the parameters whose numerical values are given above substantially influence the quantity $\sigma_r(b, t)$. The methods presented for the solution enable an analogous computation to be made in each specific case. Also the σ_φ, σ_z, ε_r, ε_φ and u can be computed at any other point of the cylinder-shell system.

REFERENCES

1. IL'YUSHIN A.A. and POBEDRYA B.E., Principles of the Mathematical Theory of Thermoviscoelasticity, Nauka, Moscow, 1970.
2. MOSKVITIN V.V., Resistance of viscoelastic Materials as it Applies to Solid Fuel Rocket Motor Charges. Nauka, Moscow, 1972.
3. TALYBLY L.KH., On the non-linear theory of temperature stresses in viscoelastic bodies, Mekhanika Komposit. Materialov, 4, 1983.
4. IL'YUSHIN A.A. and OGIBALOV P.M., Quasilinear theory of viscoelasticity and the small parameter method, Mekhanika Polimerov, 2, 1966.
5. TOKAREV O.B., Dynamic loading of a non-linearly viscoelastic cylinder with variable inner radius, Vestnik Moskhov, Gosud. Univ., Ser. Matematika, Mekhanika, 4, 1970.
6. TRICOMI F., Integral Equations, Izd. Ionstr. Lit., Moscow, 1960.
7. TALYBLY L.KH., On a method of taking account of the influence of temperature on the mechanical properties of physically linear viscoelastic media, Proceedings of the Conf. on Applied Math. and Mechanics in Kirovabad, Elm, Baku, 1984.
8. SCHAPERY R.A., An approximate method of stress analysis for a large class of problems in viscoelasticity. Purdue Univ. School Aeronaut. and Eng. Sci., Rept. A and ES, 62-12, 1963.
9. ACHENBACH J.D., Dynamic behaviour of a long viscoelastic cylinder fastened to a housing, AIAA Jnl., 4, 1965.

Translated by M.D.F.